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Deconstruction of gauge symmetry breaking by discrete symmetry and G^N unification

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Abstract. We deconstruct the non-supersymmetric $SU(5)$ breaking by discrete symmetry on the space-time $M^4 \times S^1$ and $M^4 \times S^1/(Z_2 \times Z_2')$ in the Higgs mechanism deconstruction scenario. Also we explain the subtle point of how to exactly match the continuum results with the latticized results on the quotient space S^1/Z_2 and $S^1/(Z_2 \times Z_2')$. We also propose an effective deconstruction scenario and discuss the gauge symmetry breaking by the discrete symmetry on the theory space in this approach. As an application, we suggest the G^N unification where G^N is broken down to $SU(3) \times SU(2) \times U(1)^{n-3}$ by the bifundamental link fields and the doublet–triplet splitting can be achieved.

1 Introduction

Grand unified theory (GUT) gives us a simple and elegant understanding of the quantum numbers of quarks and leptons, and the success of gauge coupling unification in the minimal supersymmetric standard model strongly supports this idea. Although the grand unified theory at a high energy scale has been widely accepted, there are some problems in GUT: the grand unified gauge symmetry breaking mechanism, the doublet–triplet splitting problem, and the proton decay problem, etc.

Recently, a new scenario proposed to address the above questions in GUT has been discussed extensively [1,2]. The key point is that the GUT gauge symmetry exists in 5 or higher dimensions and is broken down to the 4-dimensional $N = 1$ supersymmetric standard model like gauge symmetry for the zero modes due to the discrete symmetries in the brane neighborhoods or on the extra space manifolds, which become non-trivial constraints on the multiplets and gauge generators in GUT [2]. Attractive models have been constructed explicitly where the supersymmetric 5-dimensional and 6-dimensional GUT models are broken down to the 4-dimensional $N = 1$ supersymmetric $SU(3) \times SU(2) \times U(1)^{n-3}$ model, where n is the rank of the GUT group, through the compactification on various orbifolds and manifolds. The GUT gauge symmetry breaking and doublet–triplet splitting problems have been solved neatly by the discrete symmetry projections. Other interesting phenomenological issues, like μ problems, gauge coupling unifications, non-supersymmetric GUT, gauge–Higgs unification, proton decay, etc., have also been discussed [1,2].

On the other hand, deconstruction was proposed about one year ago [3]. Deconstruction is interesting because it provides a UV completion of the higher dimensional theories. A lot of phenomenological and formal issues in deconstruction scenarios have been discussed. These include the extensions of the standard model, gaugino mediated supersymmetry breaking, low energy unification, GUT breaking, electroweak symmetry breaking, anomaly inflow, the description of little string theories in terms of gauge theory, a single gauge group description of extra dimensions in the limit of a large number of colors, models with non-commutative geometry, warped background geometry, topological objects, Seiberg–Witten curves, and even the deconstruction of time and gravity, etc. [4–8]. By the way, the arrays of gauge theories, where an infinite number of gauge theories are linked by scalars, were discussed previously [9].

In this paper, we would like to discuss the deconstruction of gauge symmetry breaking by the discrete symmetry on an extra space manifold. If we know how to deconstruct those higher dimensional theories, we may have available a lot of good features in deconstruction scenarios, for example, the gauge symmetry breaking, the doublet–triplet splitting, suppressing the proton decay, the gauge–Higgs unification. (In the deconstruction language, bifundamental fields and $SU(2)_L$ Higgs unification.) For simplicity, we do not consider supersymmetry. First, we consider the Higgs mechanism deconstruction scenario in which the gauge bosons obtain the masses via the VEVs of the bifundamental link Higgs fields. We shall deconstruct the non-supersymmetric $SU(5)$ breaking by the discrete symmetry on the space-time $M^4 \times S^1$ and $M^4 \times S^1/(Z_2 \times Z_2')$, where

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M^4 is the 4-dimensional Minkowski space-time. We also explain the subtle point on how to exactly match the continuum results with the latticized results on the quotient space S^1/Z_2 and $S^1/(Z_2 \times Z_2')$. In addition, it seems to us that the Higgs mechanism deconstructions of the gauge symmetry breaking by discrete symmetry on the space-time $M^4 \times S^1$ and $M^4 \times S^1/(Z_2 \times Z_2')$ might not be the real deconstructions. The key point is that the bifundamental field U_i, which is the Schwinger line integral along the fifth dimension, should be considered as the gauge field A_5. However, the mass spectrum and the 5-dimensional wave functions for the KK modes of A_5 cannot match those for U_i in the Higgs mechanism deconstruction scenario by counting the massless modes[1]. Therefore, we propose the effective deconstruction scenario where we add the mass term for each field by hand which comes from the latticization of the kinetic term of that field along the fifth dimension. In the effective deconstruction scenario, similar to the gauge symmetry breaking by the discrete symmetry on the extra space manifold, we can define the discrete symmetry on the theory space and discuss the gauge symmetry breaking. Moreover, we show that the continuum results match the effective deconstruction results exactly. As an application, we discuss the G^N unification where G^N is broken down to $SU(3) \times SU(2) \times U(1)^{n-3}$ (n is the rank of group G) by the bifundamental link fields and the doublet–triplet splitting can be achieved. With the general G^N unification, we wish we can solve the tough problems in the traditional 4-dimensional GUT models.

Let us explain our terminology. The exact match between the nth KK mode of a bulk field in the continuum case and the corresponding state in the deconstruction case means that the nth mass eigenvalue and eigenvector of the field in the deconstruction scenario are the same as the mass and 5-dimensional wave function of the nth KK mode of the corresponding bulk field when $N >> n$.

2 Higgs mechanism deconstruction of $SU(5)$ breaking on $M^4 \times S^1$ by Wilson line

In this section, we would like to deconstruct the non-supersymmetric $SU(5)$ breaking on the space-time $M^4 \times S^1$ by a Wilson line in the Higgs mechanism deconstruction scenario. Also we want to point out that one can discuss any other GUT groups similarly because the fundamental group of S^1 is Z, i.e., $\pi_1(S^1) = Z$ [2].

2.1 $SU(5)$ breaking on $M^4 \times S^1$ by a Wilson line

Let us consider the 5-dimensional space-time which can be factorized into a product of the ordinary 4-dimensional Minkowski space-time M^4 and the circle S^1. The corresponding coordinates are x^μ ($\mu = 0, 1, 2, 3$), $y \equiv x^5$, and the radius for the fifth dimension is R. The gauge fields are denoted $A_M(x^\mu, y)$ where $M = 0, 1, 2, 3, 5$. Because $Z_2 \subset \pi_1(S^1)$, we can define the Z_2 parity operator P for a generic bulk multiplet $\Phi(x^\mu, y)$:

$$\begin{aligned}\Phi(x^\mu, y) &\to \Phi(x^\mu, y + 2\pi R) \\ &= \eta_\Phi P^{l_\Phi} \Phi(x^\mu, y)(P^{-1})^{m_\Phi}, \end{aligned} \quad (1)$$

where $\eta_\Phi = \pm 1$ and $P^2 = 1$. By the way, if the gauge group G is $SU(5)$, for a 5-plet Φ in the fundamental representation, $l_\Phi = 1$ and $m_\Phi = 0$, and for a 24-plet Φ in the adjoint representation, $l_\Phi = 1$ and $m_\Phi = 1$.

Denoting the field ϕ with parity $P = \pm$ by $\phi_\pm$, we obtain the KK mode expansions

$$\phi_+(x^\mu, y) = \sum_{n=-\infty}^{+\infty} \phi_+^n(x^\mu) \mathrm{e}^{\mathrm{i}\frac{ny}{R}}, \quad (2)$$

$$\phi_-(x^\mu, y) = \sum_{n=-\infty}^{+\infty} \phi_-^n(x^\mu) \mathrm{e}^{\mathrm{i}\frac{(n+1/2)y}{R}}. \quad (3)$$

Now let us discuss the $SU(5)$ breaking. Under parity P, the gauge fields A_M transform as

$$A_M(x^\mu, y + 2\pi R) = P A_M(x^\mu, y) P^{-1}. \quad (4)$$

Also, we choose the following matrix representation for parity operator P, which is expressed in the adjoint representation of $SU(5)$:

$$P = \mathrm{diag}(-1, -1, -1, +1, +1). \quad (5)$$

So, upon invoking P parity, the gauge generators T^A where $A = 1, 2, ..., 24$ for $SU(5)$ are separated into two sets: T^a are the gauge generators for the standard model gauge group, and $T^{\hat{a}}$ are the other broken gauge generators:

$$P\ T^a\ P^{-1} = T^a\ ,\ P\ T^{\hat{a}}\ P^{-1} = -T^{\hat{a}}. \quad (6)$$

The masses for A_M^a and $A_M^{\hat{a}}$ are n/R and $(n + 1/2)/R$, respectively. In addition, if we add a pair of Higgs 5-plets H_u and H_d in the bulk, then for each 5-plet, the doublet mass is n/R and the triplet mass is $(n + 1/2)/R$ if $\eta_{H_u} = \eta_{H_d} = +1$. In short, for the zero modes, the gauge group $SU(5)$ is broken down to $SU(3) \times SU(2) \times U(1)$, and we can solve the doublet–triplet splitting problem. The parities and masses of the fields in the $SU(5)$ gauge and Higgs multiplets are given in Table 1.

2.2 Higgs mechanism deconstruction

We consider the $SU(5)^{N+1}$ gauge theory with bifundamental fields U_i as follows:

[1] One might consider the axial gauge $A_5 = 0$ in the 5-dimensional theory. However, only part of the scalars from the U_i fields are eaten by the massive gauge bosons after the gauge symmetry breaking, and most of the physical scalars from the U_i fields can obtain masses via the Higgs mechanism. Therefore, the 5-dimensional gauge field A_5 in the axial gauge ($A_5 = 0$) cannot match the bifundamental fields U_i in the Higgs mechanism deconstruction scenario, too

	$SU(5)_0$	$SU(5)_1$	$SU(5)_2$	$\cdots$	$SU(5)_{N-1}$	$SU(5)_N$
U_0	$\Box$	$\overline{\Box}$	1	$\cdots$	1	1
U_1	1	$\Box$	$\overline{\Box}$	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
U_{N-1}	1	1	1	$\cdots$	$\Box$	$\overline{\Box}$
U_N	$\overline{\Box}$	1	1	$\cdots$	1	$\Box$

. (7)

The effective action is

$$S = \int \mathrm{d}^4x \sum_{i=0}^{N} \Big(- \frac{1}{4g^2} \mathrm{Tr} F_i^2 + \mathrm{Tr}[(D_\mu U_i)^\dagger D^\mu U_i] + \ldots \Big), \tag{8}$$

where the covariant derivative is $D^\mu U_i \equiv \partial_\mu U_i - \mathrm{i} A_\mu^i U_i + \mathrm{i} U_i A_\mu^{i+1}$ and the dots represent the higher dimensional operators that are irrelevant at low energies.

The bifundamental fields U_i obtain the vacuum expectation values (VEV) either from a suitable renormalizable potential or from some strong interactions. In order to obtain the deconstruction of the Wilson line gauge symmetry breaking, we choose the following VEVs for U_i:

$$\langle U_i \rangle = \mathrm{diag}(v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}), \quad \text{for } i = 0, 1, 2, ..., N-1, \tag{9}$$

$$\langle U_N \rangle = \mathrm{diag}(-v/\sqrt{2}, -v/\sqrt{2}, -v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}). \tag{10}$$

We would like to explain the VEV for U_N, which is different from the VEV for U_i where $i = 0, 1, 2, ..., N-1$. In general, we can take the VEV for U_N to be the same as that for U_i. The masses for the gauge bosons are given by the mass terms

$$\frac{1}{2} \sum_{i=0}^{N} g^2 v^2 (A_\mu^{\beta(i+1)} - A_\mu^{\beta i})^2,$$

where $A_\mu^{\beta(N+1)} = A_\mu^{\beta 0}$ for $\beta = a$ and $A_\mu^{\beta(N+1)} = -A_\mu^{\beta 0}$ for $\beta = \hat{a}$ due to the Wilson line gauge symmetry breaking (see Table 1). However, in the Higgs mechanism deconstruction scenario, we take $A_\mu^{\beta(N+1)} = A_\mu^{\beta 0}$ for $\beta = a$ and $\beta = \hat{a}$, so the last term in the above mass terms is $\frac{1}{2} g^2 v^2 (A_\mu^{\beta 0} - A_\mu^{\beta N})^2$ for $\beta = a$, and $\frac{1}{2} g^2 v^2 (A_\mu^{\beta 0} + A_\mu^{\beta N})^2$ for $\beta = \hat{a}$. This correct mass term can be obtained by choosing the suitable VEV for U_N, which is (10) in our model. By the way, we emphasize that the VEV for U_N in (10) is similar to the matrix representation for the parity operator P in (5), which breaks the $SU(5)$ gauge symmetry.

Thus, the $(N+1) \times (N+1)$ mass matrix for the standard model gauge boson or the column vector $(A_\mu^{a0}, A_\mu^{a1}, A_\mu^{a2}, ..., A_\mu^{aN})$ is

$$M^2_{\mathrm{SM}} = g^2 v^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ & & & \cdots & \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{11}$$

Table 1. Parity assignment and masses of the fields in the $SU(5)$ gauge and Higgs multiplets for the model with $SU(5)$ breaking by a Wilson line. The index a labels the unbroken $SU(3) \times SU(2) \times U(1)$ gauge generators, while $\hat{a}$ labels the other broken SU(5) gauge generators. The indices D and T are for doublet and triplet, respectively

P	Field	Mass ($n = 0, \pm 1, \pm 2, ...$)
$+$	A_μ^a, A_5^a, H_u^{D}, H_d^{D}	n/R
$-$	$A_\mu^{\hat{a}}$, $A_5^{\hat{a}}$ H_u^{T}, H_d^{T}	$(n+1/2)/R$

The mass spectrum or eigenvalue is

$$M_n^2 = 4 g^2 v^2 \sin^2 \left(\frac{n\pi}{N+1} \right), \tag{12}$$

where $-N/2 \leq n \leq N/2$ or $n = 0, 1, 2, ..., N$, and the corresponding nth eigenvector is

$$\alpha_n = (\alpha_n^0, \alpha_n^1, ..., \alpha_n^N), \tag{13}$$

where

$$\alpha_n^j = \frac{1}{\sqrt{N+1}} \exp \left(\mathrm{i} \frac{2jn\pi}{N+1} \right), \quad j = 0, 1, ..., N. \tag{14}$$

Noticing that $R = (N+1)/(gv)$, we find that the nth eigenvector matches the 5-dimensional wave function $(\mathrm{e}^{\mathrm{i}\frac{ny}{R}})$ of the nth KK mode for A_μ^a in the last subsection, and for $n << N$, the nth mass (eigenvalue) matches the mass of the nth KK mode for A_μ^a. So they exactly match.

The mass matrix for the non-standard model gauge boson or the column vector $(A_\mu^{\hat{a}0}, A_\mu^{\hat{a}1}, A_\mu^{\hat{a}2}, ..., A_\mu^{\hat{a}N})$ is

$$M^2_{\mathrm{NSM}} = g^2 v^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & +1 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ & & & \cdots & \\ +1 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{15}$$

The mass spectrum or eigenvalue is

$$M_n^2 = 4 g^2 v^2 \sin^2 \left(\frac{(n+1/2)\pi}{N+1} \right), \tag{16}$$

where $-N/2 \leq n \leq N/2$ or $n = 0, 1, 2, ..., N$. Also, the corresponding nth eigenvector is

$$\alpha_n = (\alpha_n^0, \alpha_n^1, ..., \alpha_n^N), \tag{17}$$

where

$$\alpha_n^j = \frac{1}{\sqrt{N+1}} \exp \left(\mathrm{i} \frac{j(2n+1)\pi}{N+1} \right), \quad j = 0, 1, ..., N. \tag{18}$$

Noticing that $R = (N+1)/(gv)$, we see that the nth eigenvector matches the 5-dimensional wave function ($e^{i\frac{(n+1/2)y}{R}}$) of the nth KK mode for $A_\mu^{\hat{a}}$ in the last subsection, and for $n << N$, the nth mass (eigenvalue) matches the mass of the nth KK mode for $A_\mu^{\hat{a}}$. Thus, they exactly match, and we want to emphasize that there is no massless mode for $A_\mu^{\hat{a}i}$.

In addition, we can discuss the deconstruction of the Higgs fields H_u and H_d. We assume that under the ith gauge group $SU(5)_i$, there is a pair of Higgs 5-plets H_u^i and H_d^i, where $i = 0, 1, 2, ..., N$. We consider the following potential:

$$V = 2g^2 \sum_{i=0}^{N} (|U_i(H_u^i)^\dagger|^2 + |H_u^{i+1}U_i|^2 + |U_i H_d^i|^2 + |(H_d^{i+1})^\dagger U_i|^2) - \sqrt{2} g^2 v \sum_{i=0}^{N} (H_u^{i+1} U_i (H_u^i)^\dagger + (H_d^{i+1})^\dagger U_i H_d^i + \text{H.C.}), \quad (19)$$

where for simplicity we define $H_u^{N+1} \equiv H_u^0$ and $H_d^{N+1} \equiv H_d^0$. The above potential for a pair of Higgs 5-plets H_u^i and H_d^i comes from the deconstruction of $(D_5 H_u)^\dagger D_5 H_u$ and $(D_5 H_d)^\dagger D_5 H_d$ in 5-dimensional theory, and the coefficients are determined by the normalization which is compatible with that of the gauge fields. In short, we see that the mass matrix for the doublet $H_u^{i\text{D}}$ or $H_d^{i\text{D}}$ is the same as that for the standard model gauge boson in (11), and the mass matrix for the triplet $H_u^{i\text{T}}$ or $H_d^{i\text{T}}$ is the same as that for the non-standard model gauge boson in (15). Therefore, similar to the discussions for A_μ^a and $A_\mu^{\hat{a}}$, the deconstruction results match the continuum results in Table 1. Then we solve the doublet–triplet splitting problem.

Furthermore, by counting the number of massless modes, we can prove that the bifundamental fields U_i cannot match the gauge fields A_5 because the U_i are Higgs fields. The correct deconstruction of the A_5 fields should have 12 massless modes, but the U_i field will have at least $12(2N+1)$ massless modes that are the Goldstone bosons and give masses to the longitudinal components of the massive gauge bosons.

In short, the gauge group is broken down to $SU(3) \times SU(2) \times U(1)$ for the zero modes, and the deconstruction results match the continuum results except for A_5 and U_i.

3 Higgs mechanism deconstruction of $SU(5)$ breaking on $M^4 \times S^1/(Z_2 \times Z_2')$

In this section, we would like to discuss the non-supersymmetric $SU(5)$ breaking on the space-time $M^4 \times S^1/(Z_2 \times Z_2')$. By the way, the $SU(5)$ breaking on the space-time $M^4 \times S^1/Z_2$ can be discussed similarly.

3.1 $SU(5)$ breaking on $M^4 \times S^1/(Z_2 \times Z_2')$

Our convention is similar to that in Sect. 2.1. The orbifold $S^1/(Z_2 \times Z_2')$ is obtained by S^1 moduloing the following equivalent classes:

$$y \sim -y, \; y' \sim -y', \quad (20)$$

where y' is defined as $y' \equiv y - \pi R/2$.

For a generic bulk multiplet $\Phi(x^\mu, y)$ which fills a representation of the gauge group G, we can define two parity operators P and P' for the Z_2 and Z_2' symmetries, respectively:

$$\Phi(x^\mu, y) \to \Phi(x^\mu, -y) = \eta_\Phi P^{l_\Phi} \Phi(x^\mu, y)(P^{-1})^{m_\Phi}, \quad (21)$$

$$\Phi(x^\mu, y') \to \Phi(x^\mu, -y') = \eta'_\Phi (P')^{l_\Phi} \Phi(x^\mu, y')(P'^{-1})^{m_\Phi}, \quad (22)$$

where $\eta_\Phi = \pm 1$ and $\eta'_\Phi = \pm 1$.

Denoting the field ϕ with $(P, P') = (\pm, \pm)$ by $\phi_{\pm\pm}$, we obtain the KK mode expansions

$$\phi_{++}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{\delta_{n,0}}\pi R}} \phi_{++}^{(2n)}(x^\mu) \cos \frac{2ny}{R}, \quad (23)$$

$$\phi_{+-}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{+-}^{(2n+1)}(x^\mu) \cos \frac{(2n+1)y}{R}, \quad (24)$$

$$\phi_{-+}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{-+}^{(2n+1)}(x^\mu) \sin \frac{(2n+1)y}{R}, \quad (25)$$

$$\phi_{--}(x^\mu, y) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{--}^{(2n+2)}(x^\mu) \sin \frac{(2n+2)y}{R}. \quad (26)$$

The 4-dimensional fields $\phi_{++}^{(2n)}$, $\phi_{+-}^{(2n+1)}$, $\phi_{-+}^{(2n+1)}$ and $\phi_{--}^{(2n+2)}$ acquire masses $2n/R$, $(2n+1)/R$, $(2n+1)/R$ and $(2n+2)/R$ upon compactification. Zero modes are contained only in the ϕ_{++} fields; thus, the matter content of the massless sector is smaller than that of the full 5-dimensional multiplet. Moreover, only the ϕ_{++} and ϕ_{+-} fields have non-zero values at $y = 0$, and only the ϕ_{++} and ϕ_{-+} fields have non-zero values at $y = \pi R/2$.

Under parity P, the gauge fields A_M transform as

$$A_\mu(x^\mu, -y) = P A_\mu(x^\mu, y) P^{-1}, \quad (27)$$

$$A_5(x^\mu, -y) = -P A_5(x^\mu, y) P^{-1}. \quad (28)$$

Under parity P' the gauge field transformations are similar to those under P.

We choose the following matrix representations for the parity operators P and P' that are expressed in the adjoint representation of $SU(5)$:

$$P = \text{diag}(+1, +1, +1, +1, +1), \quad P' = \text{diag}(-1, -1, -1, +1, +1). \quad (29)$$

So under P' parity, the $SU(5)$ gauge generators T^A where $A = 1, 2, ..., 24$ for $SU(5)$, are separated into two sets: the T^a are the gauge generators for the standard model gauge group, and the $T^{\hat{a}}$ are the other broken gauge generators:

$$P\ T^a\ P^{-1} = T^a,\ P\ T^{\hat{a}}\ P^{-1} = T^{\hat{a}}, \tag{30}$$

$$P'\ T^a\ (P')^{-1} = T^a,\ P'\ T^{\hat{a}}\ (P')^{-1} = -T^{\hat{a}}. \tag{31}$$

Therefore, the masses for the gauge fields A^a_μ, $A^{\hat{a}}_\mu$, A^a_5 and $A^{\hat{a}}_5$ are $2n/R$, $(2n+1)/R$, $(2n+2)/R$ and $(2n+1)/R$, respectively. For the zero modes, the gauge group is $SU(3) \times SU(2) \times U(1)$. Including the KK modes, the gauge groups at $y = 0$ and $y = \pi R/2$ are $SU(5)$ and $SU(3) \times SU(2) \times U(1)$, respectively.

Moreover, assuming that there exists a pair of Higgs 5-plets H_u and H_d in the bulk and $\eta_{H_u} = \eta_{H_d} = +1$, we obtain the result that for each 5-plet, the doublet mass is $2n/R$, and the triplet mass is $(2n+1)/R$. So we solve the doublet–triplet splitting problem. The parities and masses of the fields in the $SU(5)$ gauge and Higgs multiplets are given in Table 2.

3.2 Higgs mechanism deconstruction

We consider the $SU(5)^N \times (SU(3) \times SU(2) \times U(1))$ gauge theory with bifundamental fields U_i $(i = 0, 1, ..., N-1)$, U_c and U_w as follows:

	$SU(5)_0$	$SU(5)_1$	$SU(5)_2$	$\cdots$	$SU(5)_{N-1}$	$SU(3)$	$SU(2)$	$U(1)_Y$
U_0	$\square$	$\overline{\square}$	1	$\cdots$	1	1	1	0
U_1	1	$\square$	$\overline{\square}$	$\cdots$	1	1	1	0
U_2	1	1	$\square$	$\cdots$	1	1	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
U_c	1	1	1	$\cdots$	$\square$	$\overline{\square}$	1	$\frac{1}{3}$
U_w	1	1	1	$\cdots$	$\square$	1	$\square$	$-\frac{1}{2}$

(32)

This kind of models has been discussed recently in [5, 7].

The effective action is

$$S = \int \mathrm{d}^4x \sum_{i=0}^{N-2} \left(-\frac{1}{4g^2} \mathrm{Tr} F_i^2 + \mathrm{Tr}[(D_\mu U_i)^\dagger D^\mu U_i] + ... \right)$$
$$- \frac{1}{4g^2} \mathrm{Tr} F_c^2 + \mathrm{Tr}[(D_\mu U_c)^\dagger D^\mu U_c] + ...$$
$$- \frac{1}{4g^2} \mathrm{Tr} F_w^2 + \mathrm{Tr}[(D_\mu U_w)^\dagger D^\mu U_w] + ... \tag{33}$$

Because we consider the Higgs mechanism deconstruction of $SU(5)$ breaking on the space-time $M^4 \times S^1/(Z_2 \times Z_2')$, at the GUT scale, we should take the same gauge couplings for all the gauge groups $SU(5)^N \times (SU(3) \times SU(2) \times U(1))$.

We choose the following VEVs for U_i, U_c and U_w:

$$\langle U_i \rangle = \mathrm{diag}(v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}),\quad \text{for } i = 0, 1, 2, ..., N-1, \tag{34}$$

Table 2. Parity assignment and masses of the fields $(n \geq 0)$ in the $SU(5)$ gauge and Higgs multiplets for the model with $SU(5)$ breaking on $M^4 \times S^1/(Z_2 \times Z_2')$

(P, P')	Field	Mass $(n = 0, 1, 2, ...)$
$(+,+)$	A^a_μ, H^{D}_u, H^{D}_d	$2n/R$
$(+,-)$	$A^{\hat{a}}_\mu$, H^{T}_u, H^{T}_d	$(2n+1)/R$
$(-,+)$	$A^{\hat{a}}_5$	$(2n+1)/R$
$(-,-)$	A^a_5	$(2n+2)/R$

$$\langle U_c \rangle = \mathrm{diag}(v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}),$$
$$\langle U_w \rangle = \mathrm{diag}(v/\sqrt{2}, v/\sqrt{2}). \tag{35}$$

The $(N+1) \times (N+1)$ mass matrix for the standard model gauge boson is

$$M^2_{\mathrm{SM}} = g^2 v^2 \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}. \tag{36}$$

However, the mass spectrum and eigenvector of the above mass matrix do not exactly match those of A^a_μ in the last subsection because of the fixed point. This subtle point is similar to that in the brane models on $M^4 \times S^1/Z_2$ or $M^4 \times R^1/Z_2$ where the brane tension at a fixed point on the quotient space is half of that on the covering space [10]. Also, on the covering space S^1, the mass matrix for the standard model gauge boson is similar to that in (11); the continuum results and the deconstruction results do exactly match. In short, the nth column eigenvector α_n and eigenvalue M^2_n should satisfy the following equation:

$$(M^2_{\mathrm{SM}})_{ij} \alpha^j_n = \left[1 - \frac{1}{2}(\delta_{0i} + \delta_{Ni})\right] M^2_n \alpha^i_n. \tag{37}$$

The mass spectrum or eigenvalue is

$$M^2_n = 4g^2 v^2 \sin^2\left(\frac{n\pi}{2N}\right), \tag{38}$$

where $n = 0, 1, 2, ..., N$. The corresponding nth eigenvector is

$$\alpha_n = (\alpha^0_n, \alpha^1_n, ..., \alpha^N_n), \tag{39}$$

where

$$\alpha^j_n = \frac{\sqrt{2}}{\sqrt{N+1}} \cos\left(\frac{2jn\pi}{2N}\right),\quad j = 0, 1, ..., N. \tag{40}$$

Noticing that $R = 2N/(gv)$, we find that the nth eigenvector matches the 5-dimensional wave function $(\cos(2ny/R))$

of the nth KK mode for A_μ^a in the last subsection (ϕ_{++}), and for $n << N$, the nth mass (eigenvalue) matches the mass of the nth KK mode for A_μ^a in the last subsection.

The $N \times N$ mass matrix for the non-standard model gauge boson is

$$M_{\rm NSM}^2 = g^2 v^2 \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \quad (41)$$

Similarly, the mass spectrum and eigenvector of the above matrix do not match those of $A_\mu^{\hat{a}}$ in the last subsection due to the fixed point. On the covering space S^1, the mass matrix is reducible and the irreducible mass matrix is similar to that in (50). Thus, the continuum results and the deconstruction results do exactly match on the covering space S^1. In short, the nth eigenvector α_n and eigenvalue M_n^2 should satisfy the following equation:

$$(M_{\rm NSM}^2)_{ij} \alpha_n^j = \left[1 - \frac{1}{2}\delta_{0i}\right] M_n^2 \alpha_n^i, \quad (42)$$

where $j = 0, 1, 2, ..., N-1$. The mass spectrum or eigenvalue is

$$M_n^2 = 4g^2 v^2 \sin^2\left(\frac{(2n+1)\pi}{4N}\right), \quad (43)$$

where $n = 0, 1, 2, ..., N-1$. The corresponding nth eigenvector is

$$\alpha_n = (\alpha_n^0, \alpha_n^1, ..., \alpha_n^{N-1}), \quad (44)$$

where

$$\alpha_n^j = \frac{\sqrt{2}}{\sqrt{N}} \cos\left(\frac{j(2n+1)\pi}{2N}\right), \quad j = 0, 1, ..., N-1. \quad (45)$$

Noticing that $R = 2N/(gv)$, we see that the nth eigenvector matches the 5-dimensional wave function ($\cos((2n + 1)y/R)$) of the nth KK mode for $A_\mu^{\hat{a}}$ in the last subsection (ϕ_{+-}), and for $n << N$, the nth mass (eigenvalue) matches the mass of the nth KK mode for $A_\mu^{\hat{a}}$ in the last subsection.

In addition, we would like to discuss the deconstruction of the 5-dimensional fields ϕ_{-+} and ϕ_{--} because we will have this kind of field expansions when we discuss the effective deconstruction scenario in the next section. So let us give the mass matrix, eigenvalues and eigenvectors here.

The $N \times N$ mass matrix for the deconstruction of the 5-dimensional field ϕ_{-+} is

$$M_{-+}^2 = g^2 v^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}. \quad (46)$$

The mass spectrum or eigenvalue is

$$M_n^2 = 4g^2 v^2 \sin^2\left(\frac{(2n+1)\pi}{4N}\right), \quad (47)$$

where $n = 0, 1, 2, ..., N-1$. The corresponding nth eigenvector is

$$\alpha_n = (\alpha_n^1, \alpha_n^2, ..., \alpha_n^N), \quad (48)$$

where

$$\alpha_n^j = \frac{\sqrt{2}}{\sqrt{N}} \sin\left(\frac{j(2n+1)\pi}{2N}\right), \ j = 1, 2, ..., N. \quad (49)$$

The $(N-1) \times (N-1)$ mass matrix for the deconstruction of the 5-dimensional field ϕ_{--} is

$$M_{--}^2 = g^2 v^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \quad (50)$$

Because the field ϕ_{--} vanishes at the fixed point, we would like to emphasize that the mass spectrum and eigenvector for the physical field are the eigenvalue and eigenvector of the above mass matrix. The mass spectrum or eigenvalue is

$$M_n^2 = 4g^2 v^2 \sin^2\left(\frac{(n+1)\pi}{2N}\right), \quad (51)$$

where $n = 0, 1, 2, ..., N-2$. The corresponding nth eigenvector is

$$\alpha_n = (\alpha_n^1, \alpha_n^2, ..., \alpha_n^{N-1}), \quad (52)$$

where

$$\alpha_n^j = \frac{\sqrt{2}}{\sqrt{N}} \sin\left(\frac{j(n+1)\pi}{N}\right), \quad j = 1, 2, ..., N-1. \quad (53)$$

Moreover, we can discuss the deconstruction of the Higgs fields H_u and H_d. We assume that under the ith gauge group $SU(5)_i$, there is a pair of Higgs 5-plets H_u^i and H_d^i where $i = 0, 1, 2, ..., N-1$, and for the Nth gauge group, there is a pair of Higgs doublets $H_u^{\rm ND}$ and $H_d^{\rm ND}$. We consider the following potential:

$$
\begin{aligned}
V = & 2g^2 \sum_{i=0}^{N-2} (|U_i(H_u^i)^\dagger|^2 + |H_u^{i+1}U_i|^2 + |U_i H_d^i|^2 \\
& +|(H_d^{i+1})^\dagger U_i|^2) \\
& -\sqrt{2}g^2 v \sum_{i=0}^{N-2} (H_u^{i+1}U_i(H_u^i)^\dagger + (H_d^{i+1})^\dagger U_i H_d^i + \text{H.C.}) \\
& +2g^2(|U_w(H_u^{(N-1)D})^\dagger|^2 + |H_u^{\rm ND}U_w|^2 \\
& +|U_w H_d^{(N-1)D}|^2 + |(H_d^{\rm ND})^\dagger U_w|^2) \\
& -\sqrt{2}g^2 v(H_u^{\rm ND}U_w(H_u^{(N-1)D})^\dagger + (H_d^{\rm ND})^\dagger U_w H_d^{(N-1)D} \\
& +\text{H.C.}). \qquad (54)
\end{aligned}
$$

This potential for a pair of Higgs 5-plets H_u^i and H_d^i comes from the deconstruction of $(D_5 H_u)^\dagger D_5 H_u$ and $(D_5 H_d)^\dagger D_5 H_d$ in 5-dimensional theory, and the coefficients are determined by the normalization which is compatible with that of the gauge fields. So we see that the mass matrix for the doublet $H_u^{i\rm D}$ or $H_d^{i\rm D}$ is the same as that for the standard model gauge boson in (36), and the mass matrix for the triplet $H_u^{i\rm T}$ or $H_d^{i\rm T}$ is the same as that for the non-standard model gauge boson in (41). Therefore, similar to the discussions for A_μ^a and $A_\mu^{\hat{a}}$, the deconstruction results match the continuum results in Table 2 exactly. Then we solve the doublet–triplet splitting problem.

Furthermore, by counting the number of massless modes, we can prove that the bifundamental fields U_i cannot match the gauge fields A_5 because the U_i are Higgs fields. The correct deconstruction of the A_5 fields should have 0 massless modes, but the U_i field will have at least $24N$ massless modes that are the Goldstone bosons and give masses to the longitudinal components of massive gauge bosons.

In short, the gauge group is broken down to $SU(3) \times SU(2) \times U(1)$ for the zero modes, and the deconstruction results match the continuum results except for A_5 and U_i.

4 Gauge symmetry breaking by the discrete symmetry on the theory space in the effective deconstruction scenario

It seems to us that the Higgs mechanism deconstructions of the gauge symmetry breaking by discrete symmetry on the space-time $M^4 \times S^1$ or $M^4 \times S^1/(Z_2 \times Z_2')$ in the last two sections might not be the real deconstructions. The key point is that the bifundamental field U_i, which is the Schwinger line integral along the fifth dimension, should be considered as the field A_5. However, the mass spectrum and the 5-dimensional wave function for the KK modes of A_5 cannot match those for U_i in the Higgs mechanism deconstruction scenario in the above two sections by counting the massless modes. In order to have an exact match, we propose the effective deconstruction scenario and discuss the gauge symmetry breaking by the discrete symmetry on the theory space.

Before we propose the effective deconstruction scenario, let us consider the 5-dimensional $SU(5)$ theory with a pair of Higgs 5-plets H_u and H_d on the space-time $M^4 \times S^1$. If we latticized the fifth dimension with $N+1$ sites, we obtain the following mass terms for the fields A_M^i, H_u^i, H_d^i from the latticized kinetic terms of the fields along the fifth dimension, i.e., $\partial_5\phi\partial^5\phi$ for a generic bulk field $\phi(x^\mu, y)$:

$$
\begin{aligned}
V = & \sum_{i=0}^{N-1} \left(\frac{N+1}{2\pi R}\right)^2 \left(\frac{1}{2}(A_M^{i+1} - A_M^i)^2 + |H_u^{i+1} - H_u^i|^2 \right. \\
& \left. +|H_d^{i+1} - H_d^i|^2\right) \\
& + \left(\frac{N+1}{2\pi R}\right)^2 \left(\frac{1}{2}(A_M^N - \Gamma A_M^0 \Gamma^{-1})^2 \right. \\
& \left. +|H_u^N - \eta_{H_u}\Gamma H_u^i|^2 + |H_d^{i+1} - \eta_{H_d}\Gamma^{-1}H_d^i|^2\right), \qquad (55)
\end{aligned}
$$

where the subscript M denotes $0, 1, 2, 3, 5$ (μ and 5), Γ is a 5×5 matrix and is a generator of the Z_n group, which is a subgroup of $\pi_1(S^1) = Z$, i.e., $\Gamma^n = 1$. Of course, there exist some other terms, but we are not interested in them here. It is not hard for one to prove that the 5-dimensional $SU(5)$ gauge theory on $M^4 \times S^1$ is equivalent to the 4-dimensional $SU(5)^{N+1}$ gauge theory with the above effective potential and other terms in the large N limit, i.e., $N \to +\infty$. In addition, we would like to point out that the $SU(5)^{N+1}$ gauge theory with the above effective potential preserves only the $SU(5)/\Gamma$ gauge symmetry, where the gauge group $SU(5)/\Gamma$ is the commutant of Γ in $SU(5)$; mathematically speaking,

$$
SU(5)/\Gamma \equiv \{g \in SU(5) | g\Gamma = \Gamma g\}. \qquad (56)
$$

Of course, the Lagrangian is not $SU(5)^{N+1}$ gauge invariant, and then the theory is non-renormalizable. However, this latticized theory is correct because from the point of view of 4-dimensional effective theory, the 5-dimensional $SU(5)$ gauge theory on $M^4 \times S^1$ preserves only the $SU(5)/\Gamma$ gauge symmetry for the zero modes; the gauge symmetries for the non-zero KK modes of 5-dimensional gauge fields are completely broken, and the 5-dimensional theory is non-renormalizable.

Therefore, we can consider the $SU(5)^{N+1}$ gauge theory where in particular the gauge fields A_5 or the corresponding link fields U_i do not have the VEVs, and we introduce the above effective potential for the mass terms by hand. In this approach, we can consider the discrete symmetry on the theory space and discuss the gauge symmetry breaking. Moreover, the continuum results exactly match the deconstruction results. Because we add the mass terms by hand and our theory is an effective theory, we call this deconstruction scenario the effective deconstruction scenario. However, we would like to emphasize that the effective deconstruction scenario is the non-renormalizable theory, and how to construct a renormalizable effective deconstruction scenario deserves further study and is out of the scope of this paper.

4.1 Deconstruction of $SU(5)$ breaking on $M^4 \times S^1$ by Wilson line

Considering the $SU(5)^{N+1}$ gauge theory with the bifundamental link fields U_i given in (7) in Sect. 2.2, and assuming that there exists one pair of Higgs 5-plets H_u^i and H_d^i under the gauge group $SU(5)_i$ where $i = 0, 1, 2, ..., N$, we choose the following effective potential:

$$\begin{aligned} V = \sum_{i=0}^{N-1} \left(\frac{N+1}{2\pi R}\right)^2 & \left(\frac{1}{2}(A_\mu^{i+1} - A_\mu^i)^2 + \frac{1}{2}(U_{i+1} - U_i)^2 \right. \\ & \left. + |H_u^{i+1} - H_u^i|^2 + |H_d^{i+1} - H_d^i|^2 \right) \\ & + \left(\frac{N+1}{2\pi R}\right)^2 \left(\frac{1}{2}(A_\mu^N - \Gamma A_\mu^0 \Gamma^{-1})^2 \right. \\ & + \frac{1}{2}(U_N - \Gamma U_0 \Gamma^{-1})^2 \\ & \left. + |H_u^N - \eta_{H_u} \Gamma H_u^i|^2 + |H_d^{i+1} - \eta_{H_d} \Gamma^{-1} H_d^i|^2 \right), \quad (57) \end{aligned}$$

where

$$\Gamma = \mathrm{diag}(-1, -1, -1, +1, +1), \quad (58)$$

and $\eta_{H_u} = \eta_{H_d} = +1$. So we obtain the result that the parities of all the fields are the same as those in Sect. 2.1 (see Table 1), and the deconstruction results exactly match the continuum results.

4.2 Deconstruction of $SU(5)$ breaking on $M^4 \times S^1/(Z_2 \times Z_2')$

We consider the $SU(5)^{4N}$ gauge theory with bifundamental fields U_i as follows[2]:

	$SU(5)_0$	$SU(5)_1$	$SU(5)_2$	$\cdots$	$SU(5)_{N-1}$	$SU(5)_N$
U_0	$\square$	$\overline{\square}$	1	$\cdots$	1	1
U_1	1	$\square$	$\overline{\square}$	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
U_{N-1}	1	1	1	$\cdots$	$\square$	$\overline{\square}$

. (59)

	$SU(5)_N$	$SU(5)_{N+1}$	$SU(5)_{N+2}$	$\cdots$	$SU(5)_{2N-1}$	$SU(5)_{2N}$
U_N	$\overline{\square}$	$\square$	1	$\cdots$	1	1
U_{N+1}	1	$\overline{\square}$	$\square$	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
U_{2N-1}	1	1	1	$\cdots$	$\overline{\square}$	$\square$

. (60)

	$SU(5)_{2N}$	$SU(5)_{2N+1}$	$SU(5)_{2N+2}$	$\cdots$	$SU(5)_{3N-1}$	$SU(5)_{3N}$
U_{2N}	$\square$	$\overline{\square}$	1	$\cdots$	1	1
U_{2N+1}	1	$\square$	$\overline{\square}$	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
U_{3N-1}	1	1	1	$\cdots$	$\square$	$\overline{\square}$

. (61)

	$SU(5)_{3N}$	$SU(5)_{3N+1}$	$SU(5)_{3N+2}$	$\cdots$	$SU(5)_{4N-1}$	$SU(5)_0$
U_{3N}	$\overline{\square}$	$\square$	1	$\cdots$	1	1
U_{3N+1}	1	$\overline{\square}$	$\square$	$\cdots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	
U_{4N-1}	1	1	1	$\cdots$	$\overline{\square}$	$\square$

. (62)

On the theory space, there are $4N$ sites and $4N$ bifundamental link fields. We would like to add one pair of Higgs 5-plets H_u^i and H_d^i on the ith site. Moreover, we introduce the following effective potential for the mass terms:

$$\begin{aligned} V = \sum_{i=0}^{4N-1} \left(\frac{2N}{\pi R}\right)^2 & \left(\frac{1}{2}(A_\mu^{i+1} - A_\mu^i)^2 + \frac{1}{2}(U_i - U_{i+1})^2 \right. \\ & \left. + |H_u^{i+1} - H_u^i|^2 + |H_d^{i+1} - H_d^i|^2 \right). \quad (63) \end{aligned}$$

So the mass matrix for each field is similar to that in (11).

Now let us discuss the discrete symmetries on the theory space. We define $i' \equiv i + 3N$ for $0 \le i < N$ and $i' \equiv i - N$ for $N \le i < 4N$. The Z_2 and Z_2' symmetries on the theory space are defined by the following equivalent classes:

$$i \sim 4N - i \text{ for } Z_2, \;\; i' \sim 4N - i' \text{ for } Z_2'. \quad (64)$$

For a generic multiplet $\Phi^i(x^\mu)$ ($i = 0, 1, 2, ..., 4N$) which fills a representation of the gauge group $SU(5)_i$, we can define two parity operators P and P' for the Z_2 and Z_2' symmetries, respectively:

$$\Phi^i(x^\mu) \to \Phi^{4N-i}(x^\mu) = \eta_\Phi P^{l_\Phi} \Phi^i(x^\mu)(P^{-1})^{m_\Phi}, \quad (65)$$

$$\Phi^{i'}(x^\mu) \to \Phi^{4N-i'}(x^\mu) = \eta'_\Phi (P')^{l_\Phi} \Phi^{i'}(x^\mu)(P'^{-1})^{m_\Phi}, \quad (66)$$

where $\eta_\Phi = \pm 1$ and $\eta'_\Phi = \pm 1$.

Denoting the physical field $\tilde{\phi}$ with $(P, P') = (\pm, \pm)$ by $\tilde{\phi}_{\pm\pm}$, we obtain the physical field $\tilde{\phi}_{\pm\pm}^{(n)}(x^\mu)$ expansion in terms of the site fields $\phi^j(x^\mu)$:

$$\tilde{\phi}_{++}^{(2n)}(x^\mu) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{4N-1} \cos\left(\frac{j 2n\pi}{2N}\right) \phi^j(x^\mu), \quad (67)$$

$$\tilde{\phi}_{+-}^{(2n+1)}(x^\mu) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{4N-1} \cos\left(\frac{j(2n+1)\pi}{2N}\right) \phi^j(x^\mu), \quad (68)$$

$$\tilde{\phi}_{-+}^{(2n+1)}(x^\mu) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{4N-1} \sin\left(\frac{j(2n+1)\pi}{2N}\right) \phi^j(x^\mu), \quad (69)$$

$$\tilde{\phi}_{--}^{(2n+2)}(x^\mu) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{4N-1} \sin\left(\frac{j(2n+2)\pi}{2N}\right) \phi^j(x^\mu), \quad (70)$$

[2] For convenience to define Z_2 and Z_2' symmetries, we define half U_i as $U_i^\dagger$ in the previous sections. Because we consider the non-supersymmetric theory, there is no anomaly problem

The physical fields $\tilde{\phi}_{++}^{(2n)}$, $\tilde{\phi}_{+-}^{(2n+1)}$, $\tilde{\phi}_{-+}^{(2n+1)}$ and $\tilde{\phi}_{--}^{(2n+2)}$ acquire masses $\frac{4N}{\pi R}\sin\left(\frac{2n\pi}{4N}\right)$, $\frac{4N}{\pi R}\sin\left(\frac{(2n+1)\pi}{4N}\right)$, $\frac{4N}{\pi R}\sin\left(\frac{(2n+1)\pi}{4N}\right)$ and $\frac{4N}{\pi R}\sin\left(\frac{(2n+2)\pi}{4N}\right)$, respectively, where $n = 0, 1, 2, ..., 2N-1$. In the large N limit, or $N \to +\infty$ or $n << N$, the physical fields $\tilde{\phi}_{++}^{(2n)}$, $\tilde{\phi}_{+-}^{(2n+1)}$, $\tilde{\phi}_{-+}^{(2n+1)}$ and $\tilde{\phi}_{--}^{(2n+2)}$ acquire masses $2n/R, (2n+1)/R, (2n+1)/R$ and $(2n+2)/R$, respectively. Therefore, the deconstruction results exactly match the continuum results. In addition, zero modes are contained only in the $\tilde{\phi}_{++}^{(2n)}$ fields; thus, the matter content of the massless sector is smaller than that of the full multiplets in the theory. Moreover, only the $\tilde{\phi}_{++}^{(2n)}$ and $\tilde{\phi}_{+-}^{(2n+1)}$ fields have non-zero values at $i = 0$ and $i = 2N$, and only the $\tilde{\phi}_{++}^{(2n)}$ and $\tilde{\phi}_{-+}^{(2n+1)}$ fields have non-zero values at $i = N$ and $i = 3N$.

Under Z_2 parity P, the fields A_μ^i, U_i, H_u^i and H_d^i transform as

$$A_\mu^{4N-i}(x^\mu) = PA_\mu^i(x^\mu)P^{-1}, \quad (71)$$

$$U^{4N-i}(x^\mu) = -PU^i(x^\mu)P^{-1}, \quad (72)$$

$$H_u^{4N-i}(x^\mu) = PH_u^i(x^\mu), \quad (73)$$

$$H_d^{4N-i}(x^\mu) = P^{-1}H_d^i(x^\mu). \quad (74)$$

Under parity P', the gauge field and Higgs field transformations are similar to those under P.

We choose the following matrix representations for parity operators P and P' that are expressed in the adjoint representation of $SU(5)$:

$$P = \mathrm{diag}(+1, +1, +1, +1, +1),$$
$$P' = \mathrm{diag}(-1, -1, -1, +1, +1), \quad (75)$$

and then we obtain the result that the parities of all the fields are the same as those in Sect. 3.1 (see Table 2), and the deconstruction results exactly match the continuum results.

5 G^N unification

G^N unification has been discussed previously where each gauge group G is broken by the Higgs fields in its adjoint representation [11], or the gauge group G^N is broken by introducing more than one link Higgs field between the first two sites recently [6]. The doublet–triplet splitting are also considered in the deconstruction of $SU(5)$ on the disc D^2 [8]. In this section, we would like to briefly discuss the G^N unification where G^N is broken down to $SU(3) \times SU(2) \times U(1)^{n-3}$ by the bifundamental link fields in which n is the rank of the group G. We shall discuss the scenario with $G = SU(5)$ as an example, and similarly, one can discuss the scenario with $G = SU(6), SO(10), E_6$, etc., for $\pi_1(S_1) = Z$ [2]. The discussions for the doublet–triplet splitting are primitive here, and the natural solution to the doublet–triplet splitting problem in a supersymmetric scenario will be presented elsewhere [12].

5.1 $SU(5)_0 \times SU(5)_1$

The set-up is similar to that in Sect. 2.2 for $N = 1$. We choose the VEVs of U_0 and U_1 as

$$\langle U_0 \rangle = \mathrm{diag}(v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}), \quad (76)$$

$$\langle U_1 \rangle = \mathrm{diag}(-v/\sqrt{2}, -v/\sqrt{2}, -v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}). \quad (77)$$

The mass matrix for the standard model gauge boson is

$$M_{\mathrm{SM}}^2 = g^2v^2 \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad (78)$$

and the mass matrix for the non-standard model gauge boson is

$$M_{\mathrm{NSM}}^2 = g^2v^2 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \quad (79)$$

It is easy to check that only the standard model gauge bosons have zero modes and the non-standard model gauge bosons are massive.

Now let us discuss the doublet–triplet splitting. Suppose that H_u^i and H_d^i are in the fundamental and anti-fundamental representations of $SU(5)_i$ respectively, where $i = 0, 1$. With the following potential:

$$\begin{aligned} V = & 2g^2(|U_0(H_u^0)^\dagger|^2 + |H_u^1 U_0|^2 + |U_0 H_d^0|^2 + |(H_d^1)^\dagger U_0|^2) \\ & +2g^2(|U_1(H_u^1)^\dagger|^2 + |H_u^0 U_1|^2 + |U_1 H_d^1|^2 \\ & +|(H_d^0)^\dagger U_1|^2) \\ & -\sqrt{2}g^2v(H_u^1 U_0 (H_u^0)^\dagger + (H_d^1)^\dagger U_0 H_d^0 + \mathrm{H.C.}) \\ & -\sqrt{2}g^2v(H_u^0 U_1 (H_u^1)^\dagger + (H_d^0)^\dagger U_1 H_d^1 + \mathrm{H.C.}), \end{aligned} \quad (80)$$

we see that the mass matrix for the doublet $H_u^{i\mathrm{D}}$ or $H_d^{i\mathrm{D}}$ is the same as that for the standard model gauge boson in (78), and the mass matrix for the triplet $H_u^{i\mathrm{T}}$ or $H_d^{i\mathrm{T}}$ is the same as that for the non-standard model gauge boson in (79). Therefore, we solve the doublet–triplet splitting problem.

A possible interesting model is that we put three 5-plets ($\bar{5}$) fermions under $SU(5)_0$, and three 10-plets (10) fermions under $SU(5)_1$. Then the proton decay is suppressed at least at one loop level. Of course, there might exist an anomaly in the model. One way to avoid the anomaly is that we consider supersymmetry and three link fields that are in the fundamental and anti-fundamental representations of $SU(5)_0$ and $SU(5)_1$, respectively.

5.2 $SU(5)_0 \times SU(5)_1 \times SU(5)_2$

The set-up is similar to that in Sect. 2.2 for $N = 2$. However, we choose the following VEVs of U_0, U_1 and U_2, which can also give the correct mass matrices for the gauge boson and Higgs fields:

$$\langle U_i \rangle = \mathrm{diag}(-v/\sqrt{2}, -v/\sqrt{2}, -v/\sqrt{2}, v/\sqrt{2}, v/\sqrt{2}),$$
$$\text{for } i = 0, 1, 2. \quad (81)$$

The mass matrix for the standard model gauge boson is

$$M_{\rm SM}^2 = g^2 v^2 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \tag{82}$$

and the mass matrix for the non-standard model gauge boson is

$$M_{\rm NSM}^2 = g^2 v^2 \begin{pmatrix} 2 & +1 & +1 \\ +1 & 2 & +1 \\ +1 & +1 & 2 \end{pmatrix}. \tag{83}$$

It is easy to check that only the standard model gauge bosons have zero modes and the non-standard model gauge bosons are massive.

Now, let us discuss the doublet–triplet splitting. Suppose that H_u^i and H_d^i are in the fundamental and anti-fundamental representations of $SU(5)_i$ respectively, where $i = 0, 1, 2$. With the following potential:

$$\begin{aligned} V = {} & 2g^2(|U_0(H_u^0)^\dagger|^2 + |H_u^1 U_0|^2 + |U_0 H_d^0|^2 + |(H_d^1)^\dagger U_0|^2) \\ & 2g^2(|U_1(H_u^1)^\dagger|^2 + |H_u^2 U_1|^2 + |U_1 H_d^1|^2 + |(H_d^2)^\dagger U_1|^2) \\ & +2g^2(|U_2(H_u^2)^\dagger|^2 + |H_u^0 U_2|^2 + |U_2 H_d^2|^2 \\ & +|(H_d^0)^\dagger U_2|^2) - \sqrt{2} g^2 v (H_u^1 U_0 (H_u^0)^\dagger + H_u^2 U_1 (H_u^1)^\dagger \\ & +H_u^0 U_2 (H_u^2)^\dagger + \text{H.C.}) \\ & -\sqrt{2} g^2 v ((H_d^1)^\dagger U_0 H_d^0 + (H_d^2)^\dagger U_1 H_d^1 \\ & +(H_d^0)^\dagger U_2 H_d^2 + \text{H.C.}), \end{aligned} \tag{84}$$

we find that the mass matrix for the doublet $H_u^{i\rm D}$ or $H_d^{i\rm D}$ is the same as that for the standard model gauge boson in (82), and the mass matrix for the triplet $H_u^{i\rm T}$ or $H_d^{i\rm T}$ is the same as that for the non-standard model gauge boson in (83). Thus, we solve the doublet–triplet splitting problem.

A possible interesting model is that we put one family $(\bar{5} + 10)$ of fermions under one $SU(5)$ gauge group, and define the Z_3 symmetry on the model.

6 Discussion and conclusion

An interesting question is how to realize the Yukawa couplings in the deconstruction scenarios. If the 5-dimensional theory is non-supersymmetric, we can consider the 5-dimensional Yukawa couplings or 3-brane localized Yukawa couplings. In the deconstruction scenarios, we need to introduce the Yukawa couplings on all the sites for the first case. For the second case, we introduce the Yukawa couplings only on the corresponding particular site for that 3-brane. For example, if the 3-brane localized Yukawa couplings is on the 3-brane at $y = 0$ in the 5-dimensional theory, we only introduce the Yukawa couplings on the 0th site. In addition, if the 5-dimensional theory is supersymmetric, we can only consider the Yukawa couplings (superpotential) on the observable 3-brane at the fixed point because the 5-dimensional $N = 1$ supersymmetry is 4-dimensional $N = 2$ supersymmetry, and then the bulk Yukawa couplings are forbidden. In the deconstruction scenarios, we introduce the Yukawa couplings (superpotential) on the corresponding site for the observable 3-brane at the fixed point.

In this paper, we deconstruct the non-supersymmetric $SU(5)$ breaking by discrete symmetry on the space-time $M^4 \times S^1$ and $M^4 \times S^1/(Z_2 \times Z_2')$ in the Higgs mechanism deconstruction scenario. We explain the subtle point of how to exactly match the continuum results with the latticized results on the quotient space S^1/Z_2 and $S^1/(Z_2 \times Z_2')$. Because it seems to us that the Higgs mechanism deconstruction scenario might not be the real deconstruction, we propose the effective deconstruction scenario and discuss the gauge symmetry breaking by the discrete symmetry on the theory space in this approach. However, for simplicity, we do not consider supersymmetry and only discuss the GUT group $SU(5)$. So it is interesting to consider supersymmetry and discuss other ways of GUT breaking, for instance, $SU(6)$, $SO(10)$, and E_6, etc. Moreover, we can study the deconstruction of the extra space manifolds like the two-torus T^2, the disc D^2 and the annulus A^2, and discuss the general gauge symmetry and supersymmetry breaking by the discrete symmetry on the general theory space in the effective deconstruction scenario. It seems to us that we will have the link fields and Higgs unification, suppress the proton decay by R-symmetry, and solve the doublet–triplet splitting problem and the μ problem.

As an application, we suggest the G^N unification where G^N is broken down to $SU(3) \times SU(2) \times U(1)^{n-3}$ by the bifundamental link fields and the doublet–triplet splitting can be achieved. Furthermore, we can consider the general link fields, for example, $(10, \overline{10})$ for $SU(5)_0 \times SU(5)_1$. In short, with the general G^N unification, we wish we can solve the tough problems in the traditional 4-dimensional GUT models.

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References

1. Y. Kawamura, Prog. Theor. Phys. **103**, 613 (2000); hep-ph/9902423; hep-ph/0012125; hep-ph/0012352; G. Altarelli, F. Feruglio, hep-ph/0102301; L. Hall, Y. Nomura, hep-ph/0103125; T. Kawamoto, Y. Kawamura, hep-ph/0106163; A. Hebecker, J. March-Russell, hep-ph/0106166; hep-ph/0107039; R. Barbieri, L. Hall, Y. Nomura, hep-ph/0106190; hep-th/0107004; A.B. Kobakhidze, hep-ph/0102323; T. Li, hep-th/0107136; hep-ph/0108120; J.A. Bagger, F. Feruglio, F. Zwirner, hep-th/0107128; A. Masiero, C.A. Scrucca, M. Serone, L. Silvestrini, hep-ph/0107201; L. Hall, H. Murayama, Y. Nomura, hep-th/0107245; C. Csaki, G.D. Kribs, J. Terning, hep-ph/0107266; L. Hall, Y. Nomura, D. Smith, hep-ph/0107331; N. Maru, hep-ph/0108002; T. Asaka, W. Buchmuller, L. Covi, hep-ph/0108021; L. Hall, Y. Nomura,

T. Okui, D. Smith, hep-ph/0108071; N. Haba, T. Kondo, Y. Shimizu, T. Suzuki, K. Ukai, hep-ph/0108003; L. Hall, J. March-Russell, T. Okui, D. Smith, hep-ph/0108161; R. Dermisek, A. Mafi, hep-ph/0108139; T. Watari, T. Yanagida, hep-ph/0108152; Y. Nomura, hep-ph/0108170; Q. Shafi, Z. Tavartkiladze, hep-ph/0108247; L. Hall, J. March-Russell, T. Okui, D. Smith, hep-ph/0108161; N. Haba, hep-ph/0110164; R. Barbieri, L. Hall, Y. Nomura, hep-ph/0110102; L. Hall, Y. Nomura, hep-ph/0111068; M. Kubo, C.S. Lim, H. Yamashita, hep-ph/0111327; G. Bhattacharyya, K. Sridhar, hep-ph/0111345; H.D. Kim, J.E. Kim, H.M. Lee, hep-ph/0112094; N. Haba, T. Kondo, Y. Shimizu, hep-ph/0112132; A. Hebecker, hep-ph/0112230; N. Haba, T. Kondo, Y. Shimizu, hep-ph/0202191; T. Li, W. Liao, hep-ph/0202090; L.J. Hall, Y. Nomura, hep-ph/0202107; S. Dimopoulos, D.E. Kaplan, N. Weiner, hep-ph/0202136; K.W. Choi, H.D. Kim, Y.W. Kim, hep-ph/0202257; A. Hebecker, J. March-Russell, hep-ph/0204109
2. T. Li, hep-th/0110065; hep-th/0112255; C.S. Huang, J. Jiang, T. Li, W. Liao, hep-th/0112046
3. N. Arkani-Hamed, A.G. Cohen, H. Georgi, hep-th/0104005; C.T. Hill, S. Pokorski, J. Wang, hep-th/0104035; M.J. Strassler, hep-th/0104032
4. N. Arkani-Hamed, A.G. Cohen, H. Georgi, hep-ph/0105239; hep-th/0108089; H. Cheng, C.T. Hill, S. Pokorski, J. Wang, hep-ph/0104179; H. Cheng, C.T. Hill, J. Wang, hep-ph/0105323; C. Csaki, J. Erlich, C. Grojean, G.D. Kribs, hep-ph/0106044; H.C. Cheng, D.E. Kaplan, M. Schmaltz, W. Skiba, hep-ph/0106098; P.H. Chankowski, A. Falkowski, S. Pokorski, hep-ph/0109272; N. Arkani-Hamed, A.G. Cohen, D.B. Kaplan, A. Karch, L. Motl, hep-th/0110146; I. Rothstein, W. Skiba, hep-th/0109175; M. Alishahiha, hep-th/0105153; A. Adams, M. Fabinger, hep-th/0111079; K. Sfetsos, hep-th/0106126; C.T. Hill, hep-th/0109068; C. Csaki, J. Erlich, V.V. Khoze, E. Poppitz, Y. Shadmi, Y. Shirman, hep-th/0110188; M. Bander, hep-th/0107130; W. Skiba, D. Smith, hep-ph/0201056; Z. Berezhiani, A. Gorsky, I.I. Kogan, hep-th/0203016; R. Sekhar Chivukula, Hong-Jian He, hep-ph/0201164; N. Arkani-Hamed, A.G. Cohen, T. Gregoire, J.G. Wacker, hep-ph/0202089; K. Lane, hep-ph/0202093; A. Falkowski, C. Grojean, S. Pokorski, hep-ph/0203033; Z. Chacko, E. Katz, E. Perazzi, hep-ph/0203080; T. Kobayashi, N. Maru, K. Yoshioka, hep-ph/0110117
5. C. Csaki, G.D. Kribs, J. Terning, hep-ph/0107266
6. H. Cheng, K.T. Matchev, J. Wang, hep-ph/0107268
7. N. Weiner, hep-ph/0106097
8. E. Witten, hep-ph/0201018
9. M.B. Halpern, W. Siegel, Phys. Rev. D **11**, 2967 (1975)
10. T. Li, Phys. Lett. B **478**, 307 (2000); Nucl. Phys. B **582**, 176 (2000)
11. A. Davidson, K.C. Wali, Phys. Rev. Lett. **58**, 2623 (1987); P. Cho, Phys. Rev. D **48**, 5331 (1993); R.N. Mohapatra, Phys. Lett. B **379**, 115 (1996)
12. T. Li, W. Liao, in preparation